Lagrange multiplier examples; intro to multiple integrals

Note Title

Today: • 2 examples of Lagrange multipliers
• a first look at multiple integrals

Example 1

Find the points on $(x+1)^2 + (y-1)^2 = 4$ that are closest to, and farthest from, the origin.

Exercise: a) draw diagram

b) make a rough estimate of the answer

What function f do we want to maximize? Distance from origin is $\sqrt{x^2+y^2}$, but it is equivalent (and easier) to minimize and/or maximize x^2+y^2.

why? The value of x that maximizes $f(x)$ also maximizes $(f(x))^2$. Same for min. Technically, this is because the 'squaring' operation is monotonic.

So, we have

$$f(x,y) = x^2 + y^2 \quad \leftarrow \text{objective}$$
$$g(x,y) = (x+1)^2 + (y-1)^2 - 4 \quad \leftarrow \text{constraint}$$

Use Lagrange multiplier :

$$\bar{\nabla} f = \langle 2x, 2y \rangle$$
$$\bar{\nabla} g = \langle 2(x+1), 2(y-1) \rangle$$

Need $\bar{\nabla} f = \lambda \bar{\nabla} g$

i.e.

$$2x = 2\lambda(x+1) \quad \text{———— (1)}$$
$$2y = 2\lambda(y-1) \quad \text{———— (2)}$$

and

$$(x+1)^2 + (y-1)^2 = 4 \quad \text{———— (3)}$$

Solve (e.g. using Wolfram Alpha:

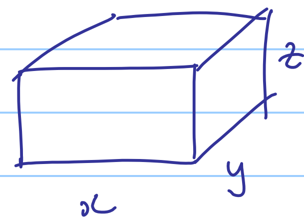
solve 2x=2*lambda*(x+1), 2y=2*lambda*(y-1), (x+1)^2 + (y-1)^2=4)

obtain two solutions: $x = -1+\sqrt{2}, \; y = 1-\sqrt{2}$ $\leftarrow$ min of f

$x = -1-\sqrt{2}, \; y = 1+\sqrt{2}$ $\leftarrow$ max of f

Example 2

Construct a box without a top. Bottom costs \$2/ sq ft. Sides cost \$3/ sq ft. Volume must be 9 cubic ft. What is minimum cost?



Cost $C(x,y,z) = 2xy + 2\times 3xz + 2\times 3yz$

$= 2xy + 6xz + 6yz$

Volume $V(x,y,z) = xyz$

Use Lagrange multipliers:

fill in as exercise

$$\vec{\nabla} C = \langle \quad , \quad , \quad \rangle$$

$$\vec{\nabla} V = \langle \quad , \quad , \quad \rangle$$

Require $\vec{\nabla} C = \lambda \vec{\nabla} V$

i.e.

$$2y + 6z = \lambda yz \qquad (1)$$

$$2x + 6z = \lambda xz \qquad (2)$$

$$6x + 6y = \lambda xy \qquad (3)$$

$$xyz = 9 \qquad (4)$$

Solve e.g. via Wolfram Alpha:

solve 2y+6z=lambda*y*z, 2x+6z=lambda*y*z, 6x+6y=lambda*x*y, xyz=9

Only real solution is: $x=3$, $y=3$, $z=1$, $\lambda=4$

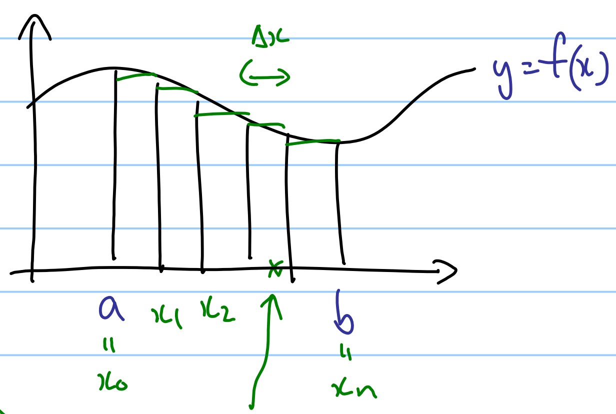
Hence minimum cost is

$$C(3,3,1) = \$54.$$

[skip the textbook section on "two constraints"]

Multiple integrals

Recall:
- one interpretation of an integral is 'area under a curve' ⟵ only makes sense if $f(x) \geq 0$
- we can estimate this area using many thin rectangles:

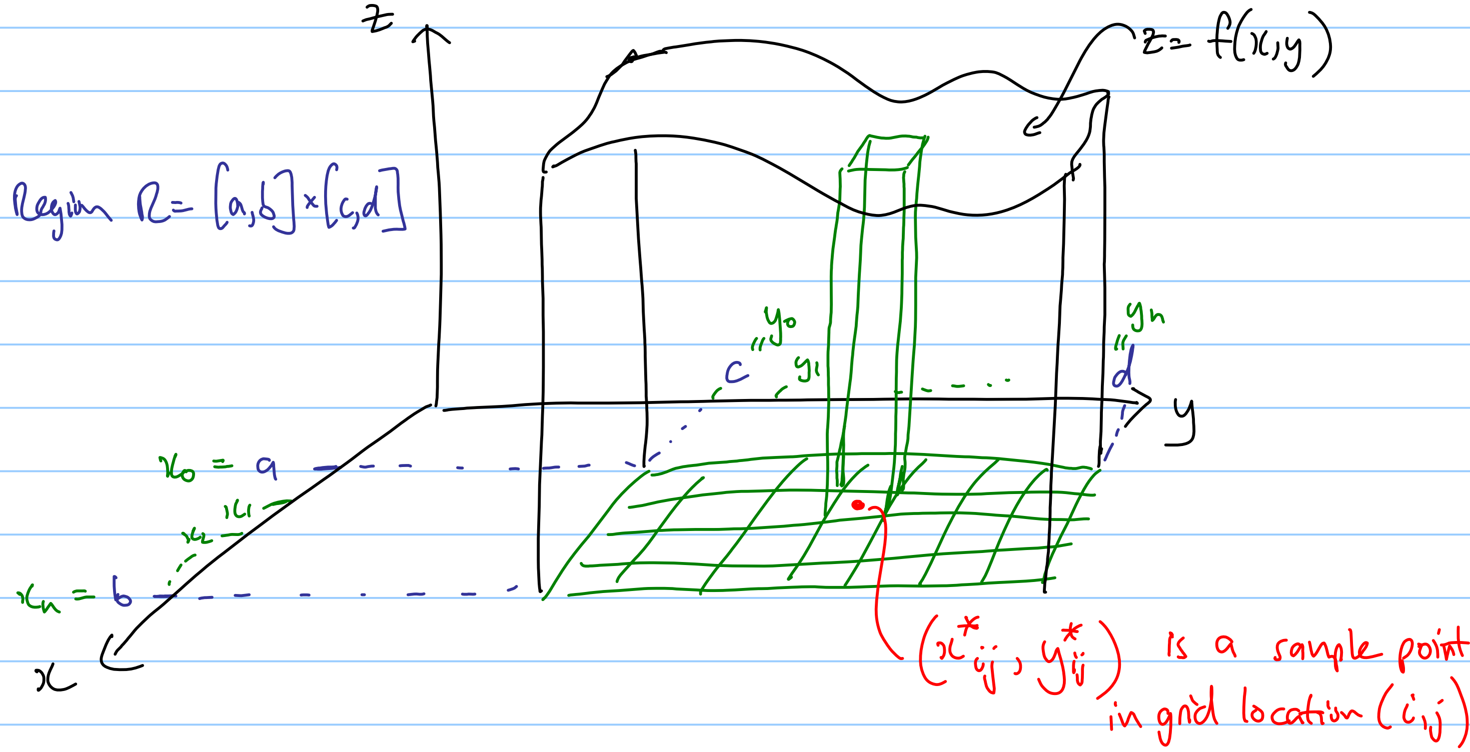


x_i^* is a sample somewhere in the ith interval

$$\text{Area} = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann sum}} = \int_a^b f(x)\,dx$$

we again need $f(x,y) \geq 0$ for this to make sense

For a function of 2 variables (i.e. a surface) we can interpret the 'volume under the surface' as an integral over both variables — i.e. a <u>double integral</u>.



$$\text{Volume } V = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*_{ij}, y^*_{ij})\, \Delta x \Delta y = \iint_R f(x,y)\, dA$$

can also be written as

$$\int_R f\, dA, \quad \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\, dx\, dy, \quad \int_{(x,y)\in R} f(x,y)\, dA, \quad \text{etc}$$

A function is <u>integrable</u> if this limit exists. It can be shown that all continuous functions are integrable.

[skip approximation in 16.1]