Pumping lemma handout

Definition of pumping

Let L be a language and w a string in L. Suppose y is a non-empty substring of w. We say y can be *pumped* in w if it can be repeated arbitrarily often while staying inside the language L. This includes the possibility that y is repeated zero times (i.e. deleted). So formally, if w can be decomposed as $w = xyz$, we say y can be *pumped* in w if

$$\text{for all } i \in \{0, 1, 2, \ldots\}, xy^iz \in L.$$

Definition of pumping before a cutoff

Sometimes we are interested in whether the substring being pumped occurs early enough in the string. That is, does y finish before the mth symbol of w? To formalize this notion, we say that if we can find any non-empty, pumpable substring y that finishes before the mth symbol of w, then w can be pumped before the cutoff m.

Note 1: To be absolutely precise, note that y can include the mth symbol of w, but can't go past it.

Note 2: This definition of "pumping before a cutoff" is not standard in the literature. It is introduced here to help our understanding of the pumping lemmma.

Informal statement of pumping lemma

IF a language is regular, THEN there is some fixed cutoff such that all strings longer than the cutoff can be pumped before the cutoff.

Informal contrapositive of pumping lemma

IF there is no fixed cutoff that allows all longer strings to be pumped before it, THEN the language isn't regular.

Or: IF, for every fixed cutoff, there is at least one longer string that can't be pumped before the cutoff, THEN the language isn't regular.

Formal statement of pumping lemma with (only) 3 quantifiers

IF L is a regular language, THEN there exists a positive integer m such that all strings in L that are longer than m can be pumped before m.

In more detail: IF L is a regular language, THEN **there exists** $m > 0$ such that **for all** $w \in L$ with $|w| \geq m$, **there exists** a decomposition $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$ (i.e. y is non-empty) such that y can be pumped.

Formal contrapositive of pumping lemma with 3 quantifiers

Let L be a language. IF it is the case that **for all** positive integers m, **there exists** a string $w \in L$ with $|w| \geq m$, such that **for all** decompositions $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$, y cannot be pumped THEN L is not regular.

Formal statement of pumping lemma with all 4 quantifiers

IF L is a regular language, THEN **there exists** $m > 0$ such that **for all** $w \in L$ with $|w| \geq m$, **there exists** x, y, z such that $w = xyz$, $|xy| \leq m$, and $|y| \geq 1$ and such that **for all** $i \in \{0, 1, 2, \ldots\}$, $xy^iz \in L$.

Formal contrapositive of pumping lemma with all 4 quantifiers

Let L be a language. IF it is the case that **for all** $m > 0$ **there exists** $w \in L$ with $|w| \geq m$, such that **for all** x, y, z with $w = xyz$, $|xy| \leq m$, and $|y| \geq 1$ **there exists** $i \in \{0, 1, 2, \ldots\}$ such that $xy^iz \notin L$, THEN L is not regular.

Rephrasing the formal contrapositive as a game

1. Opponent chooses cutoff m.
2. We choose w with $w \in L$ and $|w| \geq m$.
3. Opponent chooses decomposition. i.e. opponent selects x, y, z with $w = xyz$, $|xy| \leq m$, and $|y| \geq 1$. (Remember, when using this game in a proof, we must demonstrate we can win no matter what decomposition the opponent chooses.)
4. We choose i, and prove that $xy^iz \notin L$.
5. We win! (We have proved that L is not regular.)