A proof of Example 4.6 (p114) that more closely resembles the pumping lemma

<u>Claim</u> $L = \{ a^k b^k : k \geqslant 0 \}$ is not regular.

Proof: Assume L is regular and argue for a contradiction.
As L is regular, there is a dfa $M = \{ Q, \{a,b\}, \delta, q_0, F \}$ accepting L. Set $m = |Q| + 1$, so m is one more than the number of states in M. Consider the sequence of m states defined by $\delta^*(q_0, a^i)$, $i = 1, 2, \ldots m$.

There are more elements in this sequence than states in the dfa, so at least one state must be repeated.
[This is called the <u>pigeonhole principle</u>].
Pick one of the repeated states and call it q.
Thus, for two distinct values r and s in $\{1, 2, \ldots m\}$ we have $\delta^*(q_0, a^r) = \delta^*(q_0, a^s) = q$.

We may assume $r < s$ (otherwise swap them).

So let $\ell = s - r$. ℓ is the length of a <u>cycle</u> of 'a' links starting and ending at q. Formally,

$$\delta^*(q, a^{r+n\ell}) = q \text{ for any } n = 0, 1, 2, \ldots$$

We already know that M accepts $a^m b^m$, so

$$\delta^*(q_0, a^m b^m) = \hat{q} \text{ for some } \hat{q} \text{ in } F.$$

But we can insert a cycle of ℓ a's after the first r a's without affecting the result!

Thus

$$\delta^*(q_0, a^{m+n\ell} b^m) = \hat{q} \text{ for } n = 0, 1, 2, \ldots$$

But $\hat{q} \in F$. So in other words, M accepts $a^{m+n\ell} b^m$ for $n = 0, 1, 2, \ldots$, contradicting the fact that M accepts only elements of L.

$\square$

Briefer version you can use in EZQuiz

Assume L is regular, and a dfa $M = \{Q, \Sigma, \delta, q_0, F\}$ accepts it. Let $m = |Q| + 1$ and consider $\delta^*(q_0, a^i)$, $i = 1, \ldots m$.
By the pigeonhole principle, there exists a repeated state q in this sequence; i.e. we have r, s with $1 \leq r < s \leq m$ such that $\delta^*(q_0, a^r) = \delta^*(q_0, a^s) = q$.
Thus, a cycle of a-links of length $s-r$ starts at q,

so
$$\delta^*(q_0, a^m b^m) = \delta^*(q_0, a^{m+n(s-r)} b^m)$$

for $n = 0, 1, 2, \ldots$

This contradicts the fact that M accepts only strings of the form $a^k b^k$.

$\square$