Pumping lemma handout

Definition of pumping

Let L be a language and w a string in L. Suppose y is a non-empty substring of w. We say w can be *pumped* with y if y can be repeated arbitrarily often while staying inside the language L. This includes the possibility that y is repeated zero times (i.e. deleted). So formally, if w can be decomposed as $w = xyz$, we say w can be *pumped* with y if

$$\text{for all } i \in \{0, 1, 2, \ldots\}, xy^i z \in L.$$

Definition of pumping before a cutoff

Sometimes we are interested in whether the substring being pumped occurs early enough in the string. That is, does y finish before the mth symbol of w? To formalize this notion, we say that *w can be pumped before the cutoff m* if w can be pumped with some substring y that finishes by the mth symbol of w.

Note 1: To be absolutely precise, note that y can include the mth symbol of w, but can't go past it.

Note 2: This definition of "pumping before a cutoff" is not standard in the literature. It is introduced here to help our understanding of the pumping lemmma.

Informal statement of pumping lemma

IF an infinite language is regular, THEN there is some fixed cutoff such that all strings longer than the cutoff can be pumped before the cutoff.

Informal contrapositive of pumping lemma

For any infinite language: IF there is no fixed cutoff that allows all longer strings to be pumped before it, THEN the language isn't regular.

Or: IF, for every fixed cutoff, there is at least one longer string that can't be pumped before the cutoff, THEN the language isn't regular.

Rephrasing the contrapositive as a game

In practice, follow the rules of this game to prove that a language is not regular using the pumping lemma.

1. Opponent chooses cutoff m.
2. We choose w with $w \in L$ and $|w| \geq m$. (Make sure to choose a w that can't be pumped before m.)
3. Opponent chooses substring y. Specifically, opponent selects x, y, z with $w = xyz$, $|xy| \leq m$, and $|y| \geq 1$. (Remember, when using this game in a proof, we must demonstrate we can win no matter what substring the opponent chooses.)
4. We choose i, and prove that $xy^iz \notin L$.
5. We win! (We have proved that L is not regular.)

You must know and understand everything above this line.

Everything below this line is optional.

Formal statement of pumping lemma

IF L is an infinite regular language, THEN **there exists** $m > 0$ such that **for all** $w \in L$ with $|w| \geq m$, **there exists** x, y, z such that $w = xyz$, $|xy| \leq m$, and $|y| \geq 1$ and such that **for all** $i \in \{0, 1, 2, \ldots\}$, $xy^iz \in L$.

Formal contrapositive of pumping lemma

Let L be an infinite language. IF it is the case that **for all** $m > 0$ **there exists** $w \in L$ with $|w| \geq m$, such that **for all** x, y, z with $w = xyz$, $|xy| \leq m$, and $|y| \geq 1$ **there exists** $i \in \{0, 1, 2, \ldots\}$ such that $xy^iz \notin L$, THEN L is not regular.